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Star-factorization of symmetric complete bipartite digraphs

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Abstract

We show that a necessary and sufficient condition for the existence of an S_k-factorization of the symmetric complete bipartite digraph $K^*_{m,n}$ is $m = n \equiv 0 \pmod{k(k-1)}$.

1. Introduction

Let $K^*_{m,n}$ denote the symmetric complete bipartite digraph, and let S_k denote the orientation of the star $K_{1,k-1}$ in which all arcs are directed away from the center. A spanning subgraph F of $K^*_{m,n}$ is called an S_k-factor if each component of F is isomorphic to S_k. If $K^*_{m,n}$ is expressed as an arc-disjoint sum of S_k-factors, then this sum is called an S_k-factorization of $K^*_{m,n}$. In this paper, it is shown that a necessary and sufficient condition for the existence of such a factorization is $m = n \equiv 0$ $\pmod{k(k-1)}$.

C_k-factorizations of $K_{m,n}$ and $K^*_{m,n}$ have been completely solved by Enomoto et al. [4] and Ushio [11]. S_k-factorization of $K_{m,n}$ has been partially solved by Ushio and Tsuruno [9], Wang [12] and Du [2]. P_k-factorizations of $K_{m,n}$ and $K^*_{m,n}$ have been partially solved by Ushio [7, 10] and Du [3]. $K_{p,q}$-factorization of $K_{m,n}$ has been partially solved by Martin [6]. For graph theoretical terms, see [1, 5].

2. S_k-factor of $K^*_{m,n}$

The following theorem is on the existence of S_k-factors of $K^*_{m,n}$.

Theorem 1. *$K^*_{m,n}$ has an S_k-factor if and only if* (i) $m + n \equiv 0 \pmod{k}$, (ii) $(k-1)n - m \equiv 0 \pmod{k(k-2)}$, (iii) $(k-1)m - n \equiv 0 \pmod{k(k-2)}$, (iv) $m \leqslant k-1)n$ *and* (v) $n \leqslant (k-1)m$.

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Proof. Suppose that $K^*_{m,n}$ has an S_k-factor F. Let t be the number of components of F. Then $t = (m+n)/k$. Hence, Condition (i) is necessary. Among these t components, let t_1 and t_2 be the numbers of components whose centers are in V_1 and V_2, respectively. Then, since F is a spanning subgraph of $K^*_{m,n}$, we have $t_1 + (k-1)t_2 = m$ and $(k-1)t_1 + t_2 = n$. Hence, $t_1 = ((k-1)n - m)/k(k-2)$ and $t_2 = ((k-1)m - n)/k(k-2)$. From $0 \leqslant t_1 \leqslant m$ and $0 \leqslant t_2 \leqslant n$, we must have $m \leqslant (k-1)n$ and $n \leqslant (k-1)m$. Conditions (ii)–(v) are, therefore, necessary.

For those parameters m and n satisfying (i)–(v), let $t_1 = ((k-1)n - m)/k(k-2)$ and $t_2 = ((k-1)m - n)/k(k-2)$. Then t_1 and t_2 are integers such that $0 \leqslant t_1 \leqslant m$ and $0 \leqslant t_2 \leqslant n$. Hence, $t_1 + (k-1)t_2 = m$ and $(k-1)t_1 + t_2 = n$. Using t_1 vertices in V_1 and $(k-1)t_1$ vertices in V_2, consider $t_1 S_k$'s whose endvertices are in V_2. Using the remaining $(k-1)t_2$ vertices in V_1 and the remaining t_2 vertices in V_2, consider t_2 S_k's whose endvertices are in V_1. Then these $t_1 + t_2$ S_k's are arc-disjoint and they form an S_k-factor of $K^*_{m,n}$. □

Corollary 2. *$K^*_{n,n}$ has an S_k-factor if and only if $n \equiv 0 \pmod{k}$.*

3. S_k-factorization of $K^*_{m,n}$

We use the following notation.

Notation. Given an S_k-factorization of $K^*_{m,n}$, let r be the number of factors, b be the total number of components, t be the number of components of each factor, t_1 and t_2 be the numbers of components whose centers are in V_1 and V_2, respectively, among t components of each factor. For a vertex x in V_i, let $r_i(x)$ and $s_i(x)$ be the numbers of components in which x is a center and an endvertex, respectively.

We give the following necessary condition for the existence of an S_k-factorization of $K^*_{m,n}$.

Theorem 3. *If $K^*_{m,n}$ has an S_k-factorization then $m = n \equiv 0 \pmod{k(k-1)}$.*

Proof. Suppose that $K^*_{m,n}$ has an S_k-factorization. Then $b = 2mn/(k-1)$, $t = (m+n)/k$, $r = b/t = 2kmn/(k-1)(m+n)$, $t_1 = ((k-1)n - m)/k(k-2)$, $t_2 = ((k-1)m - n)/k(k-2)$, $m \leqslant (k-1)n$ and $n \leqslant (k-1)m$. Moreover, $r_1(u) + s_1(u) = r$, $(k-1)r_1(u) = n$, $s_1(u) = n$, $r_2(v) + s_2(v) = r$, $(k-1)r_2(v) = m$, and $s_2(v) = m$. Therefore, $r_1(u)$ and $s_1(u)$ ($r_2(v)$ and $s_2(v)$) do not depend on $u(v)$, respectively. Thus, we have $r = r_1(u) + s_1(u) = kn/(k-1)$ and $r = r_2(v) + s_2(v) = km/(k-1)$. Therefore, $m = n$ is necessary. Moreover, when $m = n$, we have $b = 2n^2/(k-1)$, $t = 2n/k$, $r = kn/(k-1)$, $t_1 = t_2 = n/k$, $r_1 = r_2 = n/(k-1)$, $s_1 = s_2 = n$. Therefore, $n \equiv 0 \pmod{k(k-1)}$ is also necessary. □

We prove the following extension theorem, which we use later in this paper.

Theorem 4. *If $K^*_{n,n}$ has an S_k-factorization, then $K^*_{sn,sn}$ has an S_k-factorization for every positive integer s.*

Proof. Let V_1, V_2 be the independent sets of $K^*_{sn,sn}$. Divide V_1 and V_2 into s subsets of n vertices, respectively. Construct a new graph G with a vertex set consisting of the subsets which were just constructed. In this graph, two vertices are symmetrically adjacent if and only if the subsets come from disjoint independent sets of $K^*_{sn,sn}$. G is the symmetric complete bipartite digraph $K^*_{s,s}$. Noting that the cardinality of each subset identified with a vertex set of G is n and that $K^*_{s,s}$ has a $K^*_{1,1}$-factorization, we see that the desired result is obtained. □

We use the following notation for sequences.

Notation. Let A and B be two sequences of the same length $(k-1)^2$ such as

$$A\colon\ a_1, a_2, \ldots, a_{(k-1)^2},$$

$$B\colon\ b_1, b_2, \ldots, b_{(k-1)^2}.$$

If $b_i = ((a_i + c) \bmod k(k-1))$ $(i = 1, 2, \ldots, (k-1)^2)$, then we write $B = A + c$, where the residues $((a_i + c) \bmod k(k-1))$ are integers in the set $\{1, 2, \ldots, k(k-1)\}$.

We give the following sufficient condition for the existence of an S_k-factorization of $K^*_{n,n}$.

Theorem 5. *If $n \equiv 0 \pmod{k(k-1)}$, $K^*_{n,n}$ has an S_k-factorization.*

Proof. Put $n = k(k-1)s$. When $s = 1$, let $V_1 = \{1, 2, \ldots, k(k-1)\}$ and $V_2 = \{1', 2', \ldots, \{k(k-1)\}'\}$. Let I_C, J_E, J_C and I_E be the following sequences of length $(k-1)^2$:

$$I_C\colon\ 1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, k-1, k-1, \ldots, k-1,$$

$$J_E\colon\ 1', 2', \ldots, \{(k-1)^2\}',$$

$$J_C\colon\ \{(k-1)^2+1\}', \ldots, \{(k-1)^2+1\}', \{(k-1)^2+2\}', \ldots,$$
$$\{(k-1)^2+2\}', \ldots, \{k(k-1)\}', \ldots, \{k(k-1)\}',$$

$$I_E\colon\ k, k+1, \ldots, k(k-1)$$

and let C and E denote the sequences (I_C, J_C) and (J_E, I_E). Let c_p and e_p be the pth element of C and E, respectively $(p = 1, 2, \ldots, 2(k-1)^2)$.

Join two vertices c_p and e_p with an arc $c_p \to e_p$ ($p = 1, 2, \ldots, 2(k-1)^2$). Construct a graph F with two vertex sets $\{c_p\}$ and $\{e_p\}$ and an arc set $\{c_p \to e_p\}$. Then F is an S_k-factor of $K^*_{k(k-1),k(k-1)}$.

Construct k^2 sequences C_{ij} such that

$$C_{ij}\colon\ I_C + (j-1)(k-1),\ J_C + (i-1)(k-1) \quad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, k).$$

Construct k^2 sequences E_{ij} such that

$$E_{ij}\colon\ J_E + (i-1)(k-1),\ I_E + (j-1)(k-1) \quad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, k).$$

Construct k^2 S_k-factors F_{ij} with C_{ij} and E_{ij} ($i = 1, 2, \ldots, k; j = 1, 2, \ldots, k$). Then it is easy to show that F_{ij} are arc-disjoint and that their sum is an S_k-factorization of $K^*_{k(k-1),k(k-1)}$.

Applying Theorem 4, $K^*_{k(k-1)s,k(k-1)s}$ has an S_k-factorization for every positive integer s. □

Theorem 6. *$K^*_{m,n}$ has an S_k-factorization if and only if $m = n \equiv 0 \pmod{k(k-1)}$.*

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